Chapter 4 Scalar and vector fields

Sec 4.1 Higher order partial derivatives

2nd order partial derivatives

- Notation If $z = f(x,y)$ we write

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$\underbrace{(f_x)_y = f_{xy} = f_{12}}_{\text{L to R}} = \underbrace{\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}}_{\text{R to L}} \quad \text{and so on.}$$

Example $f(x,y) = x^3 + x^2y^3 - 2y^2$

$f_x = 3x^2 + 2xy^3$ $\qquad f_y = 3x^2y^2 - 4y$

$f_{xx} = 6x + 2y^3$ $\qquad f_{yx} = 6xy^2$

$f_{xy} = 6xy^2$ $\qquad f_{yy} = 6x^2y - 4$

We can keep going: $f_{xyx} = 6y^2$, etc.

Notice that $f_{xy} = f_{yx}$

- Clairaut's Theorem (equality of mixed partial derivatives)

Let f be a real-valued function of m variables $x_1, \dots, x_m$.
If all of f's 2nd partial derivatives are continuous, then
$f_{x_i x_j} = f_{x_j x_i}$ for all $i, j = 1, \dots, m$.

- Functions that satisfy Laplace's equation, $f_{xx} + f_{yy} = 0$, are called harmonic.

Example Check that $u(x,y) = e^x \sin y$ is harmonic.

Sec 4.3 Extreme values of real-valued functions

- **Defs** - $f(x,y)$ has a local/relative maximum at (a,b) if $f(x,y) \leq f(a,b)$ whenever (x,y) is near (a,b).
  - $f(x,y)$ has a local/relative minimum at (a,b) if $f(x,y) \geq f(a,b)$ whenever (x,y) is near (a,b).
  - $f(x,y)$ has a global/absolute maximum at (a,b) on $D \subseteq \mathbb{R}^2$ if $f(x,y) \leq f(a,b)$ for all $(x,y) \in D$.
  - $f(x,y)$ has a global/absolute minimum at $(a,b) \in$ on $D \subseteq \mathbb{R}^2$ if $f(x,y) \geq f(a,b)$ for all $(x,y) \in D$.
  - A point (a,b) in the domain of $f(x,y)$ is a critical/stationary point if either $\nabla f(a,b) = (f_x(a,b), f_y(a,b)) = \vec{0}$ or if at least one of $f_x(a,b)$, $f_y(a,b)$ doesn't exist.

- **Recall** If $\nabla f(a,b) \neq \vec{0}$ then f will:
  - increase most rapidly in the direction of $\nabla f(a,b)$.
  - decrease " " " " " " $-\nabla f(a,b)$. Thus...

- **Theorem** If f has a local max/min at (a,b), then (a,b) is a critical point.

- Graphically, this says that the tangent plane at a critical point is horizontal (if it exists).

Indeed, $z = f(a,b) + \underbrace{f_x(a,b)}_{=0}(x-a) + \underbrace{f_y(a,b)}_{=0}(y-b)$

__Example__ $f(x,y) = x^2 + y^2 - 2x - 6y + 14$

$= 4 + (x-1)^2 + (y-3)^2$

$f_x = 2x - 2 = 0 \longrightarrow x = 1$

$f_y = 2y - 6 = 0 \longrightarrow y = 3$ $\longrightarrow$ Only critical point is $(1,3)$.

Since $f(1,3) = 4$ and $f(x,y) \geq 4 + 0 + 0 = 4$ always

$f(1,3) = 4$ is a local and global minimum.

__Example__ $f(x,y) = y^2 - x^2$

$f_x = -2x$

$f_y = 2y$ $\longrightarrow$ Only critical point is $(0,0)$.

If (x,y) is near $(0,0)$ and: $x = 0$, then $f(x,y) = y^2 \geq 0$

$y = 0$, then $f(x,y) = -x^2 \leq 0$

so $f(0,0) = 0$ is neither a local min nor max.

- __Def__ A critical point that is not a local max or min is called a __saddle point__.

- __2nd Derivative Test__: Suppose the 2nd partial derivatives of $f(x,y)$ are continuous near (a,b), and $f_x(a,b) = f_y(a,b) = 0$.

Let $D = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2$ (= the determinant of the __Hessian matrix__)

- If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local max.
- If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local min.
- If $D < 0$, then $f(a,b)$ is a saddle point.
- If $D = 0$, the test is inconclusive.

Example $f(x,y) = x^3 - 3x^2 - 3y^2 + 3xy^2$

$f_x = 3x^2 - 6x + 3y^2 = 0 \qquad f_y = -6y + 6xy = 6y(-1+x) = 0$

$\rightarrow y = 0$ or $x = 1$.

If $y = 0$, then $f_x = 3x^2 - 6x = 3x(x-2) = 0 \rightarrow x = 0$ or 2.

If $x = 1$, then $f_x = -3 + 3y^2 = 0 \quad \rightarrow y = \pm 1$.

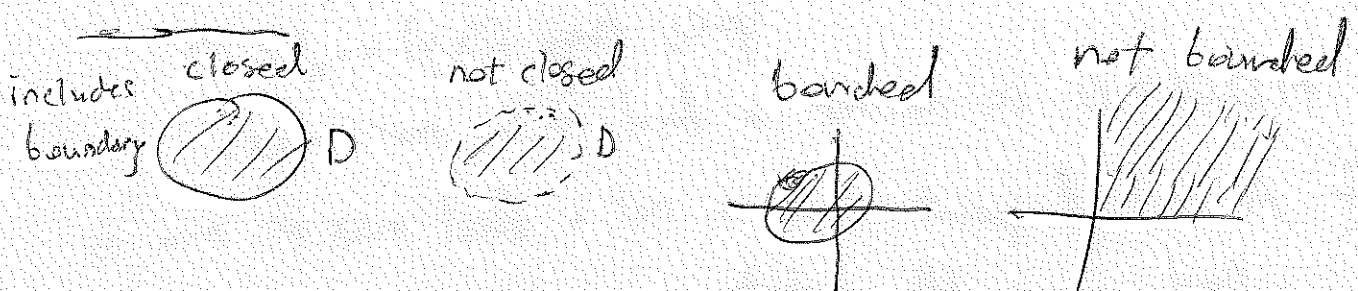
So there are 4 critical points, $(0,0), (2,0), (1,1), (1,-1)$.

We'll check $(0,0)$:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6x-6 & 6y \\ 6y & -6+6x \end{vmatrix} = (6x-6)(-6+6x) - 36y^2$$

$D(0,0) = 36 > 0$ and $f_{xx}(0,0) = -6 < 0$, so

$(0,0)$ is a local max.



- Extreme Value Theorem: A continuous function $f(x,y)$ on a closed, bounded set D attains its absolute max/min on D.

- To find the absolute max/min of a continuous function f on a closed, bounded set D:
  1. Find the values of f at the critical points in D.
  2. Find the smallest/largest values of f on the boundary of D.
  3. Take the largest/smallest values above as the absolute max/min.

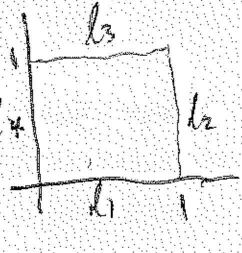
Note: In the next section will see an easier way to apply this method. For now we do it "by hands".

Example Absolute max/min of $f(x,y)=xy$ on the rectangle

$$D=\{(x,y): 0\le x\le 1,\ 0\le y\le 1\} = [0,1]\times[0,1]$$

Soln 1. $f_x = y$ $\longrightarrow$ only critical point is $(0,0)$,
$f_y = x$ which is in D, and $f(0,0)=0$.

2.



Analyzing f on $l_1,\dots,l_4$ we see that

On l_1, $f(x,y)=0$

On l_2, $0\le f(x,y)\le 1$, and $f(1,1)=1$ (only)

On l_3, $0\le f(x,y)\le 1$, and $f(1,1)=1$ (only)

On l_4, $f(x,y)=0$.

Thus: Absolute max $= f(1,1)=1$.

Absolute min of 0 at infinitely many points on l_1 and l_4.

Note For more complicated regions, we could parametrize the boundary and apply single-variable calculus. For example,

$$l_2(t)=(1,0)+t(0,1)=(1,t),\quad t\in[0,1].$$

For f restricted to l_2, we have $f(l_2(t))=f(1,t)=t$, $t\in[0,1]$.

$f'=1\neq 0$ so f has no critical points on l_2.

We check the endpoints of $[0,1]$ and confirm f has an absolute max at $t=1$.

Sec 4.4 Lagrange multipliers:

Finding the min/max of $f(x,y,z)$ subject to a constraint $g(x,y,z)=K$.

- Method of Lagrange multipliers

1. Solve the system $\nabla f = \lambda \nabla g$
   $g(x,y,z) = K$
2. Plug solutions (x,y,z) into f.

- λ is called a Lagrange multiplier.

Example: Find the max/min of $f(x,y) = 4xy$ subject to the constraint that $\underbrace{x^2+y^2}_{g(x,y)} = 4$.

Soln 1. $\nabla f = (4y, 4x) = \lambda(2x, 2y) = \lambda \nabla g$.

$4y = 2x\lambda$
$4x = 2y\lambda$ $\rightarrow \lambda = \frac{2y}{x} = \frac{2x}{y} \rightarrow 2y^2 = 2x^2 \rightarrow x^2 = y^2$
$x^2+y^2 = 4$ $\rightarrow y = \pm x$

$\rightarrow x^2 + x^2 = 2x^2 = 4 \rightarrow x = \pm\sqrt{2} \rightarrow y = \pm\sqrt{2}$

$$\left.\begin{aligned} f(\sqrt{2}, \sqrt{2}) &= 8 \\ f(-\sqrt{2}, -\sqrt{2}) &= 8 \end{aligned}\right] \text{max.}$$

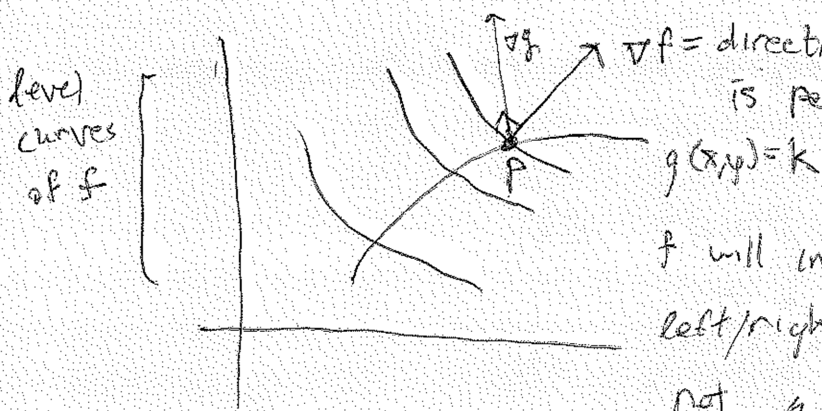
$$\left.\begin{aligned} f(\sqrt{2}, -\sqrt{2}) &= -8 \\ f(-\sqrt{2}, \sqrt{2}) &= -8 \end{aligned}\right] \text{min}$$

- λ is the approximate change in max value of f if K (from $g(x,y)=K$) is increased by 1.
  So for the max of 8 at $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$ we have
  $\lambda = \frac{2y}{x} = 2$. So if instead the constraint was $x^2+y^2 = 5$,
  then the max of $f \approx 8 + 2 = 10$.

- Why does this work?

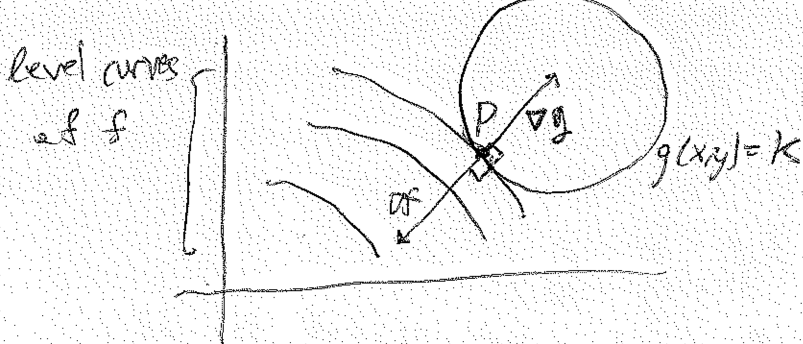
Visualize f by its level curves

$g(x,y)=K$ is the level curve of value K for $z=g(x,y)$.



∇f = direction of largest rate of increase is perpendicular to level curves.

f will increase/decrease as we move left/right along $g(x,y)=K$ so P is not a max/min.



In this case, ∇f and ∇g are parallel, so $\nabla f = \lambda \nabla g$.

If we move either way along $g(x,y)=K$, we're moving more in the direction of $-\nabla f$, so f will decrease, and P is a local max and a candidate for a global max.

- Combining this with previous absolute max/min methods:

<u>Example</u> Find the max/min of $f(x,y)=4xy$ on the disk $x^2+y^2 \le 4$.

<u>Soln</u> We need to check critical points in the disk and its boundary circle $x^2+y^2=4$. We already did the latter in the previous example.

$f_x = 4y \longrightarrow$ f only has one critical point, $(0,0)$, which is in the disk.
$f_y = 4x$

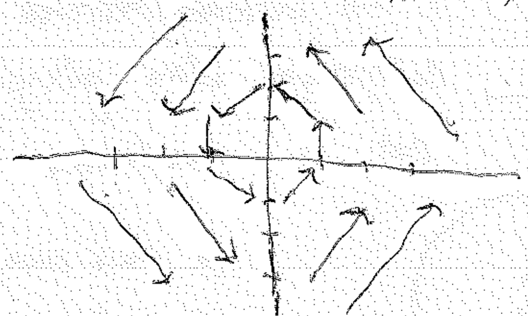
$f(0,0)=0$, so subject to the new constraint, the max/min of f remain 8 and -8.

Sec 4.6 Divergence and curl

• **Def** A function $F: U \subseteq \mathbb{R}^m \to \mathbb{R}^m$ is called a vector field (on U).
Note that the dimension m of the domain and range is the same.

We think of a vector field as assigning to a point $(x_1, \dots, x_m) \in \mathbb{R}^m$ a vector $F(x_1, \dots, x_m)$.

Example Graph the vector field $F(x,y) = (-y, x)$.



etc.

• **Def** The operator ∇ ("del") is

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

It's action on $f: \mathbb{R}^3 \to \mathbb{R}$ is the usual gradient, $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$.
(this is called the gradient vector field of f)

• **Def** It's action on $F = (F_1, F_2, F_3): \mathbb{R}^3 \to \mathbb{R}^3$ is called the divergence of F and is the dot product

$$\operatorname{div} F = \nabla \cdot F = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (F_1, F_2, F_3) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

• **Def** The curl of $F = (F_1, F_2, F_3)$ is the cross product

$$\operatorname{curl} F = \nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}.$$

Note For $F = (F_1, F_2)$, $\operatorname{curl} F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & 0 \end{vmatrix}$.

<u>Example</u> For $F(x,y) = (-y, x)$ we have

$$\operatorname{div} F = \nabla \cdot F = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\cdot(-y,x) = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) = 0 + 0 = 0,$$ and

$$\operatorname{curl} F = \nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 0 \end{vmatrix}\vec{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ -y & 0 \end{vmatrix}\vec{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ -y & x \end{vmatrix}\vec{k} = 2\vec{k}.$$

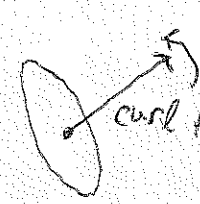
- <u>Interpretation</u> If F is the velocity vector field of a fluid, then $\operatorname{div} F$ measures the rate of expansion per unit area (volume), in $\mathbb{R}^2$ ($\mathbb{R}^3$).

  $\operatorname{div} F < 0 \longrightarrow$ fluid is compressing (sink)

  $\operatorname{div} F > 0 \longrightarrow$ fluid is expanding (source)

  $\operatorname{div} F = 0 \longrightarrow$ fluid is incompressible

and $\|\operatorname{curl} \vec{F}\|$ measures the tendency of the fluid to swirl around the axis in the direction of $\operatorname{curl} F$.



- <u>Properties</u> $\operatorname{curl}(\operatorname{grad} F) = \nabla \times \nabla f = \vec{0}$

  $\operatorname{grad}(\operatorname{curl} F) = \nabla \cdot (\nabla \times F) = 0$

- <u>Def</u> The action of the <u>Laplace operator</u> Δ (or ∇^2 or $\nabla\cdot\nabla$) on a twice differentiable real-valued function f is

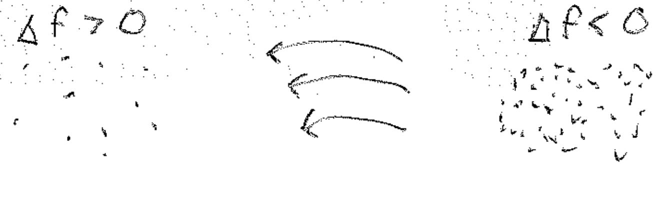
  $\Delta f = \operatorname{div}(\operatorname{grad} f) = \nabla \cdot \nabla f = f_{xx} + f_{yy} + f_{zz}$ (in $\mathbb{R}^3$).

  It's action on a twice-differentiable vector field $F = (F_1, F_2, F_3)$ is

  $\Delta F = (\Delta F_1, \Delta F_2, \Delta F_3)$.

- <u>Interpretation</u> Δ relates to the diffusion of ~~fluid~~ bacteria in air, eg:

$\Delta f > 0$ $\qquad\qquad$ $\Delta f < 0$



Sec 4.7 Implicit Function Theorem

An implicit function is determined by an equation, eg. $x^2+y^2-1=0$
We can't write this as $y =$ one expression in terms of x.

$$y = \begin{cases} \sqrt{1-x^2} & \text{top half} \\ -\sqrt{1-x^2} & \text{bottom half} \end{cases}$$

But near a certain point, we do know that $y=\sqrt{1-x^2}$, say near $(0,1)$.

The Implicit Function Theorem tells us when we can solve a system of equations for a set of variables, in terms of the others near some points.

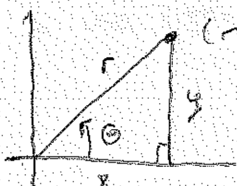
- Implicit Function Theorem (special case).

Given
$$\begin{aligned} F_1(x,y,z,u,v,w) &= 0 \\ F_2(x,y,z,u,v,w) &= 0 \\ F_3(x,y,z,u,v,w) &= 0 \end{aligned}$$
, let $(x_0, y_0, z_0, u_0, v_0, w_0) = (\vec{x}_0, \vec{u}_0)$.

If
$$\begin{vmatrix} \frac{\partial F_1}{\partial u}(\vec{x}_0,\vec{u}_0) & \frac{\partial F_1}{\partial v}(\vec{x}_0,\vec{u}_0) & \frac{\partial F_1}{\partial w}(\vec{x}_0,\vec{u}_0) \\ \frac{\partial F_2}{\partial u}(\vec{x}_0,\vec{u}_0) & \frac{\partial F_2}{\partial v}(\vec{x}_0,\vec{u}_0) & \frac{\partial F_2}{\partial w}(\vec{x}_0,\vec{u}_0) \\ \frac{\partial F_3}{\partial u}(\vec{x}_0,\vec{u}_0) & \frac{\partial F_3}{\partial v}(\vec{x}_0,\vec{u}_0) & \frac{\partial F_3}{\partial w}(\vec{x}_0,\vec{u}_0) \end{vmatrix} \neq 0,$$
then u, v, w can be solved uniquely for in terms of x, y, z) near $(\vec{x}_0, \vec{u}_0)$.

- The IFT is helpful for change of variables:

Example Polar coordinates



When can we solve $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ for r and θ?

i.e., change from Cartesian to polar coordinates.

Solution Rewrite as
$$F_1(x,y,r,\theta) = x - r\cos\theta = 0.$$
$$F_2(x,y,r,\theta) = y - r\sin\theta = 0.$$

Then $\begin{vmatrix} \frac{\partial F_1}{\partial r} & \frac{\partial F_1}{\partial \theta} \\ \frac{\partial F_2}{\partial r} & \frac{\partial F_2}{\partial \theta} \end{vmatrix} = \begin{vmatrix} -\cos\theta & r\sin\theta \\ -\sin\theta & -r\cos\theta \end{vmatrix} = r$

Thus, so long as $r \neq 0$, i.e., away from the origin.

In fact, we have the standard conversion $r = \sqrt{x^2+y^2}$

$\tan\theta = \frac{y}{x}$

(at the origin, θ is not uniquely defined)

Example $f(x,y) = x^2+y^2-1=0$

Wherever $\frac{\partial f}{\partial y} = 2y \neq 0$ we can solve for y locally.

Indeed, $y = \sqrt{1-x^2}$ near (x,y) for $y>0$

$y = -\sqrt{1-x^2}$ near (x,y) for $y<0$